Really broken numbers

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provides insights into children's thinking about fractions through their drawings and explanations.

I think that sometimes starting from a position of relative ignorance can be an advantage. I once heard an interview with a man who rode a bicycle around Australia. When asked whether he did a lot of training in preparation for the ride he said that he had rarely been on a bicycle before he set out around Australia. He claimed that if he had known how hard it was going to be he probably would never have started. I mention this by way of explaining spending the past three years of my life somewhat obsessed with the teaching of fractions. It is not as if I had never thought about the teaching of fractions before. Indeed, I must confess to having taught fractions many times since the late 1970s. My fascination with fractions grew out of the revision of the Mathematics syllabus in NSW, from Kindergarten to Year 10. I knew that many students had difficulty dealing with fractions. I also knew that the revised syllabus had increased the expectations of what students could do with fractions as well as introducing them to fractions earlier.

What went wrong?

For a long time we have known that many students experience difficulties in working with fractions (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Mack, 1995). Beyond the algorithmic manipulation of fractions lie the related difficulties of the underpinning concept. A student may be able to “invert and

multiply" to divide by a fraction and yet not be able to determine how many times you can make a recipe needing three-quarters of a cup of milk if you have three cups of milk. For many people, the concept of a fraction is not connected to the algorithmic manipulation of number pairs used to record fractions. As Kirkpatrick et al. (2001) state, "Rules for manipulating symbols are being memorized, but students are not connecting those rules to their conceptual understanding, nor are they reasoning about those rules." (p. 234)

The interpretation of fractions as parts of a whole is commonly used in teaching fractions. For example, three-eighths is frequently described in class as three parts out of eight equal parts. Experienced teachers always stress the word "equal" when talking about fractions. Students are then expected to demonstrate their understanding of three-eighths by shading in parts of a shape. For example, shade three-eighths ($\frac{3}{8}$) of each shape in Figure 1:

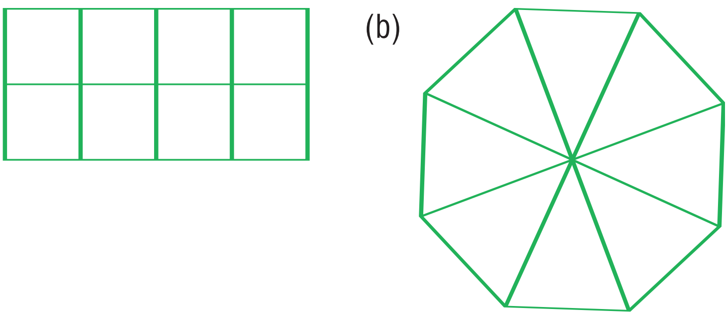

Figure 1. Shading three-eighths.

If you think about this type of question, which I claim is quite common, it is not unusual for students to come to see fractions as a kind of "double count". You count the total number of parts (just to be sure it is not a trick question) and you count and shade three of the parts. You do indeed make two whole number counts.

In a practical fraction task used to assess 11-year-old students in England, reported in Dickson, Brown and Gibson (1984), students were presented with 4 square tiles, 3 yellow and 1 red and asked, "What fraction of these squares are red?" (see Figure 2).

Only 64% of the students assessed were successful. Many of those who were incorrect gave the answer one-third, presumably losing the sense of the whole.

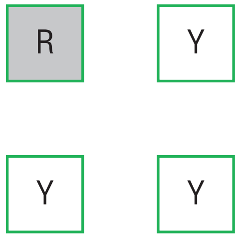

Figure 2. Discrete parts of a whole.

This process of carrying out double counts with part-whole models of fractions has some drawbacks. Kieren (1988, p. 177) clearly describes the limits of part-whole models in teaching fractions.

> Because part-whole models of fractions conveniently help produce fractional language, the school mathematics fraction language of teacher and texts alike tend to orient a student to a static double count image and knowledge of fractions. The child, while being able to produce "correct" answers to questions, develops a mental model which is inappropriately inclusive (parts of a whole), rather than a powerful measure of inclusion (comparison to a unit)…

In the dominant teaching method used to introduce fractions, students learn to divide objects into equal parts. Next, they learn to count the number of parts of interest and place the result of this count above the count of the total number of parts. This part-whole model of recording is used to introduce the tool of fraction

symbols. This in turn is followed by the traditional algorithmic manipulation of whole numbers, known as operations with fractions.

We could blame our students for being somewhat deficient in their learning of fractions, yet I am confident that students often do learn what they have been taught. A friend of mine while working with a Year 4 class told me of the following exchange. The students were shown a circle (Figure 3) and asked what fractions they could see.

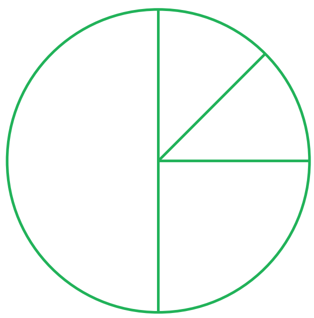

Figure 3. Divided circle.

The whole class agreed when one student said, "You have to have equal parts to have fractions, so there aren't any fractions in this circle".

The symbol system used to record fractions may be very powerful yet it is counter-intuitive. The symbol system emphasises "two whole numbers", the numerator and the denominator. For example, many students understand "a/b" as denoting a part-whole relationship, for example that "3/7" means "three out of seven" (Brown, 1993). This interpretation means that "7/3" does not make sense (Mack, 1995).

Understanding fractions

One of the most commonly stated aims of teaching mathematics is to have students "understand". Yet what do we mean when we speak of the need to understand?

Suppose that a teacher reminds a class that to find an equivalent fraction you multiply the numerator and the denominator by the same number. A student who has been absent says he does not understand, so the teacher gives him an explanation as follows:

"Say you had one-half and you wanted to find another fraction equal to one-half. If you multiply the numerator by three and the denominator by three you work out that three-sixths is the same as one-half."

Soon after watching the explanation, the student is completing an exercise. Should you ask this student if he now understands, he would most likely say that he did. Indeed, he would justify his understanding by referring to all the questions he has answered correctly. For this student, and possibly his teacher, to understand is to have a rule and to be able to use it.

This type of understanding was described by Richard Skemp (1978) as *instrumental understanding*. Many students can and do learn the algorithmic skill of operating with fractions, while being unsure of what fractions are.

In contrast to instrumental understanding is the broader notion of *relational understanding*: knowing both what to do and why. The limitations of instrumental understanding, sometimes referred to as "rules without reasons", are readily demonstrated.

Developing an instrumental understanding of mathematics is often quicker and less demanding cognitively than developing a relational understanding. Consequently, the rewards of instrumental learning are more immediate and more apparent. It will clearly result in a page of correct answers in a shorter time. Why then should we attempt to develop a relational understanding of fractions?

Emphasising only the algorithms for operating with fractions works as long as each step is correctly remembered. It is like following instructions on how to get to a certain destination, when the instructions are given in terms of left and right turns. If you follow all of the instructions correctly you will get to your destination but if you need to make even a small

detour you will soon be hopelessly lost. One false step off the instrumental path and your chances of finding the path again are slim.

The algorithms with fractions are really algorithms with whole number — you multiply, add or subtract whole numbers. It is not surprising then that many children appear to see fractions as two whole numbers — three-quarters is the whole number three written over the whole number four. When viewed this way, it is not uncommon for children to apply whole number strategies to fraction problems (Lamon, 1999; Mack, 1995; Streefland, 1993).

Look at the responses that the students chose as answers to the following task.

> Estimate the answer to $\frac{12}{13}+\frac{7}{8}$.
>
> You will not have time to solve the problem using paper and pencil.

	Percentage choosing answer	
Answer	Age 13	Age 17
1	7	8
2	14	37
19	28	21
21	27	15
I don't know	24	16

(Carpenter et al., 1981) p. 36

The responses to this item suggest that 55% and 36% respectively of 13 and 17 year-olds appear to be using a whole number approach. These choices do not make sense if students understand what the symbols mean and are reasoning about the quantities represented by the symbols. A further 14% and 16% appear to have no knowledge of how to answer the question.

Partition fractions or quantity fractions

The research into fractions has described a range of different ways of thinking about fractions (Behr, Lesh, Post & Silver, 1983; Kieren, 1976; Mack, 1990; Ohlsson, 1988). The descriptions of the characteristics or sub-constructs of fractions have included ideas such as fractions as operators (stretching and shrinking), quotients, measures, part-whole relationships, rates, and ratios. Perhaps not surprisingly, the diversity of approaches to describing fractions within the research literature has not been helpful to improving the learning of fractions. Indeed, it has been described as unsatisfactory in regard to designing instruction for an integrative understanding of fractions (Thompson & Saldanha, 2003).

Rather than canvassing the range of descriptions of fractions available in the research, I will describe a simpler way of thinking about the teaching of fractions. The Japanese approach (Yoshida, 2004) looks at *partition fractions* and *quantity fractions*. If you partition (separate or divide) objects into b parts equally and select a out of b, the amount $\frac{a}{b}$ is defined as a partition fraction (see Figure 4). Therefore, $\frac{3}{8}$ (of a pikelet) is a type of partition fraction. On the other hand, quantity fractions are defined as fractions that have a universal unit. In dealing with fractions as mathematical objects, this idea of a universal unit is very useful.

Asking the question, which is larger, one-half or three-eighths, only makes sense if the question is one of quantity fractions. The quantity fractions access a universal unit, a unique unit-whole, which is independent of any situations. If one-half and three-eighths do not refer to a universal whole, we cannot compare them. We must identify the objects that are being compared. Is one-half of a small pizza larger than three-eighths of a large pizza?

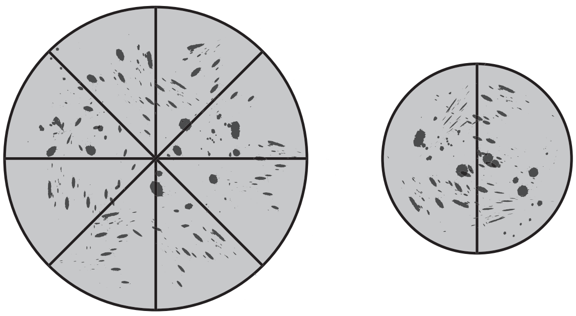

Figure 4. Partition fractions.

Part-whole relationships are characterised by partition fractions. For both discrete and continuous quantities, the unit-whole is often implied. It is difficult for students to become aware of a unit-whole when the unit-whole is often implicit in everyday situations. The essential idea of quantity fractions is that fractions themselves can express quantities.

Sharing diagrams

If the symbol system and algorithms with fractions emphasise whole numbers rather than fractions as quantities, what can we do to develop a quantitative sense of fractions? We can involve students in equi-partitioning (breaking into equal parts). I like paper folding activities because the unit-whole is not lost. Counting and colouring in pre-partitioned shapes many be a useful activity in counting, but it does nothing to significantly develop the fraction concept. We can also help students to link fractions to the process of sharing or division. The best way that I know of to tap into students' thinking about fractions is through the use of sharing diagrams. Students' recordings with "sharing diagrams" are very important as they provide both a support to their thinking and access to the strategies and partitioning students use with fractions. Look at the responses to the following question.

> Draw what would happen if I have 6 cups of milk and a recipe needs three-quarters of a cup of milk. How many times can I make the recipe before I run out of milk?

This student (Figure 5) has partitioned each cup into four pieces and has identified (by numbering) the three quarters that belong to each recipe.

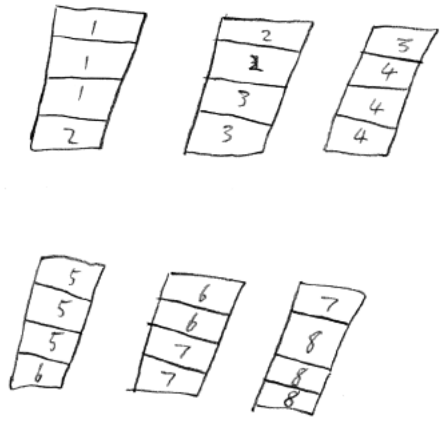

Figure 5

This student (Figure 6) has identified the three-quarters in each of the six cups and has numbered the three-quarter units. The fourth three-quarter unit has been formed by accumulating the remaining one-quarter out of each of the first three cups. Similarly, the eighth three-quarter unit is formed from the remaining one-quarter in each of the final three cups.

The diagram suggests that the student knows that three-quarters plus one-quarter is one whole.

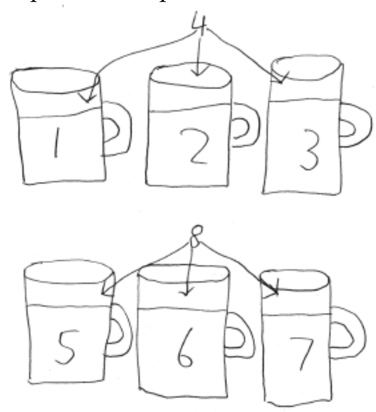

Figure 6

This student's sharing diagram (Figure 7) shows a successive accumulation of the three-quarter units. It also suggests that this student knows that three-quarters is the same as one-half and one-quarter.

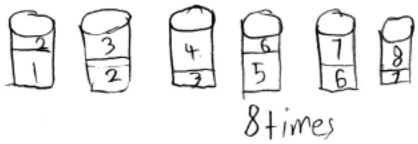

Figure 7

Using students' drawings is not a new idea. My use of sharing diagrams with fractions was influenced by the work of Marilyn Burns (1987) and Susan Empson (1995).

Contradictory beliefs

One of the things that I find fascinating arising from my work with students, is the capacity for students to hold contradictory beliefs about fractions. In looking at the answers of thousands of students in NSW, I was surprised by the ability of students to hold what I would describe as contradictory ideas about fractions. A Year 6 student when asked to shade one-third and one-sixth of a circle responded as in Figure 8.

By itself, you might consider this answer a little disconcerting. I must admit that from a broad cross-section of students in Years 4, 5, 6, 7 and 8 it is not all that uncommon. Looking at this response you would feel confident that you knew this student's misconception. However, Figure 9 shows the same student's response when asked to indicate which is the larger of two quantity fractions and to explain her reasoning. The surprising response to three of these questions involving thirds and sixths is shown in Figure 9.

Although all of the questions have the correct answers, questions 24 and 25 make no use of an equal-whole in comparing the two fractions. Perhaps even more startling is that the student is now accessing a correct image of one-third and in question 26 she now also displays a correct representa-

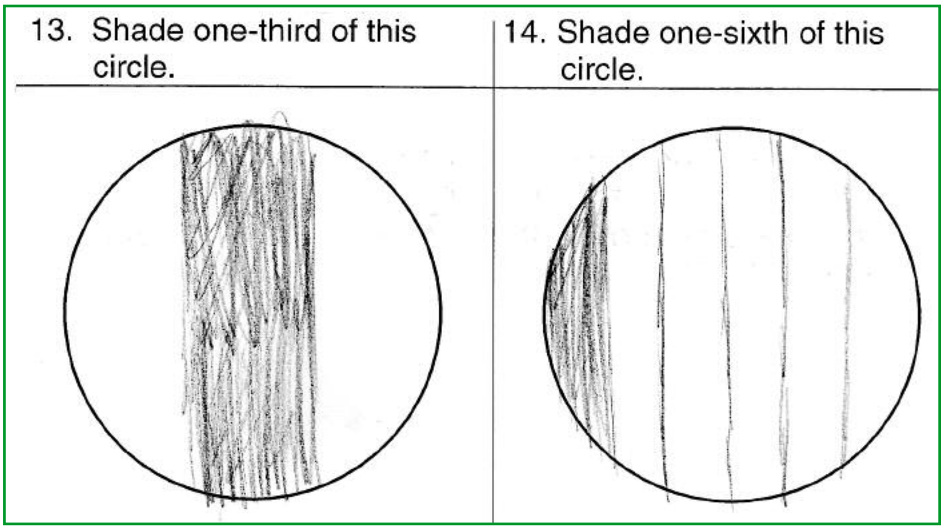

Figure 8. A Year 6 student's response.

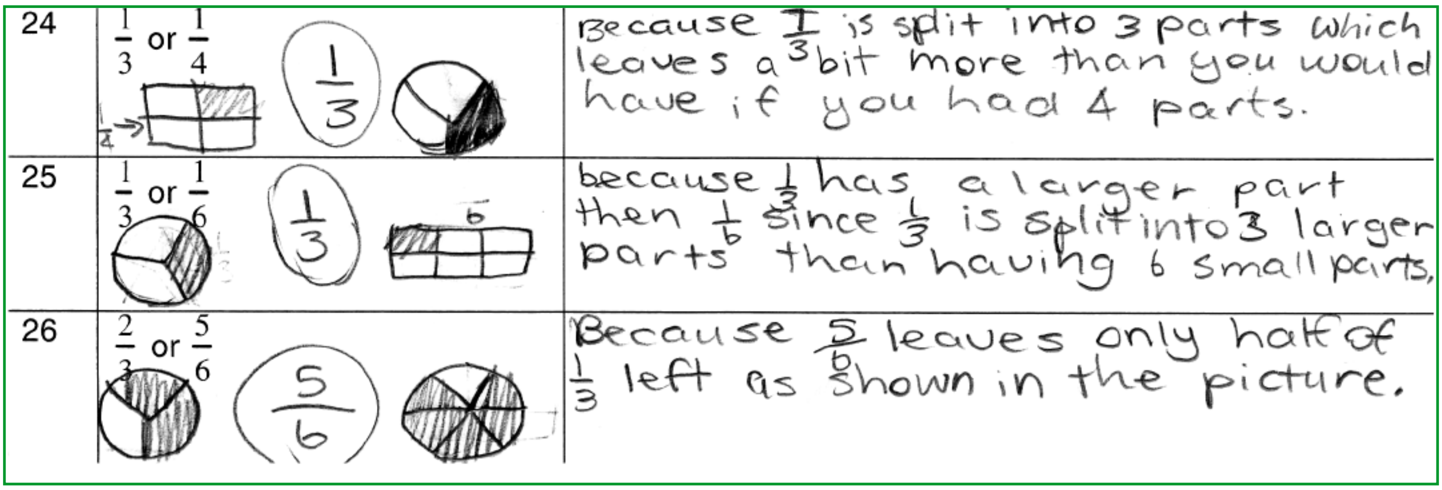

Figure 9. The same student's images of thirds and sixths.

tion of sixths and the relationship to thirds.

I take this to be an example of the difference between a consistent concept image of fractions and an *evoked* concept image. Tall and Vinner (1981) described a concept image as all of the cognitive structure in the individual's mind that is associated with a given concept, "which includes all of the mental pictures and associated properties and processes" (p. 152). The evoked concept image is the portion of the concept image activated at a particular time. In this way, seemingly conflicting images may be evoked at different times without necessarily producing any sense of conflict in a child.

The word fraction is from the Latin *frangere* (to break) and so fractions are sometimes referred to as "broken numbers". It appears that even the concept of fractions as quantities is itself broken or fractured. The concept image of fractions can hold contradictory ideas that are not in tension because they are infrequently evoked at the same time.

The fraction concept is not a single idea. Students may be proficient in one area of fractions while also holding contradictory beliefs. Consequently, the fraction concept does not respond to simple remediation methods. Students' explanations are critical in determining what they understand of fractions. The recordings students make as they reason with fractions provide us with insights into their thinking.

References

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In M. Landau (Ed.), *Acquisition of Mathematics Concepts and Processes* (pp. 91–126). New York: Academic Press.

Brown, C. A. (1993). A critical analysis of teaching rational number. In T. A. Romberg (Ed.), *Rational Numbers: An Integration of Research* (pp. 197–218). Hillsdale, NJ: Erlbaum.

Burns, M. (1987). *A Collection of Math Lessons from Grade 3 through 6*. New Rochelle, NY: Cuisenaire Company of America.

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M. & Reys, R. E. (1981). *Results from the Second Mathematics Assessment of the National Assessment of Educational Progress*. Washington, DC: National Council of Teachers of Mathematics.

Dickson, L., Brown, M. & Gibson, O. (1984). *Children Learning Mathematics: A Teacher's Guide to Recent Research*. East Sussex: Holt, Rinehart & Winston.

Empson, S. B. (1995). Using sharing situations to help children learn fractions. *Teaching Children Mathematics, 2* (2), 110–114.

Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. A. Lesh (Ed.), *Number and Measurement: Papers from a Research Workshop* (pp. 101–144). Columbus, OH: ERIC/SMEAC.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In M. J. Behr & J. Hiebert (Eds), *Number Concepts and Operations in the Middle Grades* (pp. 162–181). Reston, VA: National Council of Teachers of Mathematics.

Kirkpatrick, J., Swafford, J. & Findell, B. (2001). *Adding It Up: Helping Children Learn Mathematics*. Washington, DC.: National Academy Press.

Lamon, S. J. (1999). *Teaching Fractions and Ratios for Understanding*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*, 16–32.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26* (5), 422–441.

Ohlsson, S. (1988). Mathematical meaning and applicational meaning in the semantics of fractions and related concepts. In M. Behr (Ed.), *Number Concepts and Operations in the Middle Grades* (pp. 53–92). Reston, VA: National Council of Teachers of Mathematics.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher, 26* (3), 9–15.

Streefland, L. (1993). Fractions: A realistic approach. In T. A. Romberg (Ed.), R*ational Numbers: An Integration of Research* (pp. 289–326). Hillsdale, NJ: Lawrence Erlbaum Associates.

Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.

Thompson, P. W. & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In D. Schifter (Ed.), *A Research Companion to Principals and Standards for School Mathematics* (pp. 95–114). Reston, VA: National Council of Teachers of Mathematics.

Yoshida, K. (2004). Understanding how the concept of fractions develops: A Vygotskian perspective. In M. J. Hoines & A. B. Fugelstad (Eds), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (pp. 473–480). Bergen, Norway.

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